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INFINITELY MANY FIBRED KNOTS HAVING THE SAME ALEXANDER POLYNOMIAL

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INTRODUCTION

THE ALEXANDER polynomial is in general quite a crude invariant of a knot, as there are infinitely many knots having a given Alexander polynomial. It has been conjectured by Neuwirth[3] that this is false for fibred knots. In this paper I give an example of infinitely many fibred knots with the same Alexander polynomial.

The construction uses certain closed braids, and in the course of the proof a subsidiary result is established, showing that two closed braids are equivalent in a certain natural geometric sense if and only if the corresponding words in the braid group are conjugate.

§1. CLOSED BRAIDS

Let B_n be the braid group on n strings, see e.g. [1]. From an element $B \in B_n$ we can construct an oriented link as shown in Fig. 1 consisting of the closed braid $\hat{B}$ and an unknotted curve k, the *axis* of the closed braid.

THEOREM 1. *If there is an orientation-preserving homeomorphism of S^3 which carries the closed braid $\hat{B}$ to the closed braid $\hat{C}$, sending the axis of $\hat{B}$ to the axis of $\hat{C}$ and preserving the string orientation, then B and C are conjugate in B_n.*

Proof. Using the faithful representation of B_n on the group of automorphisms of F_n, the free group of rank n, which arise from homeomorphisms of a disc with n holes we can give a presentation of G, the fundamental group of the link $\hat{B} \cup k$, based at a point on the boundary of a tubular neighbourhood of k. We have

$$G = \{x_1, \ldots, x_n, t \colon \bar{B}(x_i) = t^{-1}x_it, i = 1, \ldots, n\},$$

where $\bar{B}$ is the automorphism corresponding to B, t is represented by the meridian of k, and $x_1, \ldots, x_n$ by loops contained in a fixed disc D which spans k and meets $\hat{B}$ in exactly n points. In this disc the loop representing x_i is freely homotopic to a simple closed curve containing just one point of $\hat{B} \cap D$, and the loops are chosen so that $x_1x_2 \ldots x_n$ is homotopic to the boundary of D, and so represents the longitude of k.

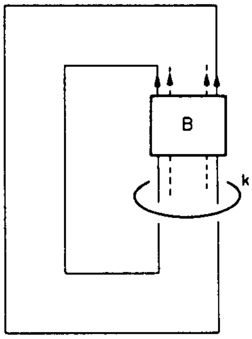

Fig. 1.

Each oriented meridian of any string of $\hat{B}$ is then freely homotopic to x_i for some i, by first sliding it along the string to lie in D.

Now let $H = \{y_1, \ldots, y_n, s;\ \bar{C}(y_i) = s^{-1}y_is,\ i = 1, \ldots, n\}$ be the fundamental group of the other link and let $\theta: H \to G$ be the isomorphism which results from the homeomorphism. We shall have $\theta(s) = t$ and $\theta(y_1y_2 \ldots y_n) = x_1x_2 \ldots x_n$, since the axes (with appropriate orientation) are preserved. Now $\theta(y_i)$ will be represented by a curve freely homotopic to a meridian of a string of $\hat{B}$, and hence by an element conjugate in G to some x_j (not x_j^{-1} since string orientation is preserved). The subgroup $K = \langle x_1, \ldots, x_n\rangle$ is normal, and freely generated by $x_1, \ldots, x_n$, so $\theta(y_i) \in K$, and since the automorphism $\bar{B}$ has the property that $\bar{B}(x_i)$ is conjugate in K to x_j, for some j, we can deduce that $\theta(y_i)$ is conjugate in K to some x_k.

Define an automorphism α of $K \cong F_n$ by $\alpha(x_i) = \theta(y_i)$. The facts that $\alpha(x_1 \ldots x_n) = x_1 \ldots x_n$ and that $\alpha(x_i)$ is conjugate in K to some x_k ensure that there is a braid $A \in B_n$ with $\bar{A} = \alpha$, see [1, p. 30]. Then

$$\begin{aligned}\alpha(\bar{C}(x_i)) &= \theta(\bar{C}(y_i))\\ &= \theta(s^{-1}y_is)\\ &= t^{-1}\alpha(x_i)t\\ &= \bar{B}(\alpha(x_i)) \text{ for each } i.\end{aligned}$$

Thus $\bar{A}\bar{C} = \bar{B}\bar{A}$, and so $C = A^{-1}BA$, since the representation of B_n is faithful.

§2. CONSTRUCTION OF FIBRED KNOTS

Convention. All homeomorphisms of S^3 or solid tori will be orientation-preserving.

Choose $B \in B_n$ such that $\hat{B}$ is the trivial knot. The complement of a tubular neighbourhood of $\hat{B}$ is then a solid torus containing the axis k of $\hat{B}$ with winding number n (the linking number of $\hat{B}$ and k). The complement of k in this solid torus fibres over S^1 having a disc with n holes as fibre. One boundary component of this disc is essentially k, and the others form n longitudes of the boundary torus.

Now choose a fibred knot l, and take a faithful, i.e. longitude-preserving, homeomorphism from the solid torus containing k to a tubular neighbourhood of l. The image of k is clearly a fibred knot, for its fibres can be constructed by piecing together n copies of the fibre of l with the discs with n holes which fibre the solid torus. This is like the construction used in [5]. The new knot has l as a companion, and its Alexander polynomial is $\Delta_l(t^n)\Delta_k(t) = \Delta_l(t^n)$, where Δ_l is the Alexander polynomial of l, and $\Delta_k(t) = 1$ since k is unknotted.

THEOREM 2. *If two knots which have been constructed as above from the fibred knot l using braids C_1 and C_2 are isotopic then there is an axis-preserving homeomorphism of S^3 carrying $\hat{C}_1$ to $\hat{C}_2$ (but not necessarily preserving string orientation). Consequently C_1 is conjugate to C_2 or its reverse, i.e. the braid with the same letters as C_2 read backwards.*

COROLLARY. *The braids $C_i = \sigma_1\sigma_2^{2i+1}\sigma_3\sigma_2^{-2i} \in B_4$, $i \geq 0$, provide an infinite sequence of fibred knots with Alexander polynomial $\Delta_l(t^4)$ for any fibred knot l.*

Proof of Corollary. Write $\phi: B_4 \to B_3$ for the homeomorphism given by $\phi(\sigma_1) = \phi(\sigma_3) = \sigma_1$, $\phi(\sigma_2) = \sigma_2$. Then $\phi(C_i) = \sigma_1\sigma_2^{2i+1}\sigma_1\sigma_2^{-2i} \in B_3$, and it is enough to prove that $\widehat{\phi(C_i)}$ is not conjugate in B_3 to $\phi(C_j)$ or its reverse if $i \neq j$. This is clear, since the closed braid $\phi(C_i)$ consists of two strings with linking number $i - 1$, and the same is true for the reverse braid. Figure 2 shows the closed braid $\hat{C}_i$, which is clearly the trivial knot, and $\widehat{\phi(C_i)}$.

Proof of Theorem 2. Suppose that a knot K in S^3 can arise from the axes of two different unknotted closed braids $\hat{C}_1$ and $\hat{C}_2$ by replacing the tubular neighbourhood of a fixed knot l with the complement of the closed braid. Then K is contained in two solid tori V_1 and V_2 in such a way that the complement of V_1 is homeomorphic to the

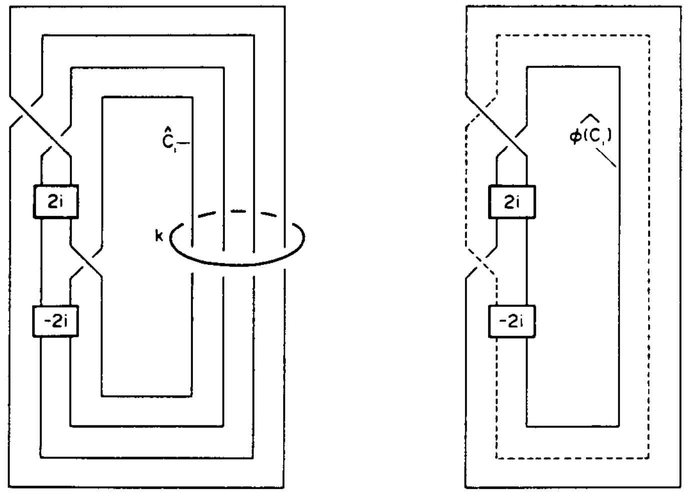

Fig. 2.

complement of V_2, and to the complement of l. Also there is a faithful homeomorphism from the complement of $\hat{C}_i$ to V_i carrying the axis to K for each I. We shall show that there is a faithful homeomorphism from V_1 to V_2 preserving K. For then there will be an axis preserving homeomorphism taking $\hat{C}_1$ to $\hat{C}_2$, as required.

Schubert[4, p. 216], shows that when a knot K lies inside solid tori V_1 and V_2 then V_2 can be isotoped, keeping K fixed, so that one of the following holds:

1. V_1 lies inside V_2.
2. V_2 lies inside V_1.
3. The closed complement of V_2 lies inside V_1.

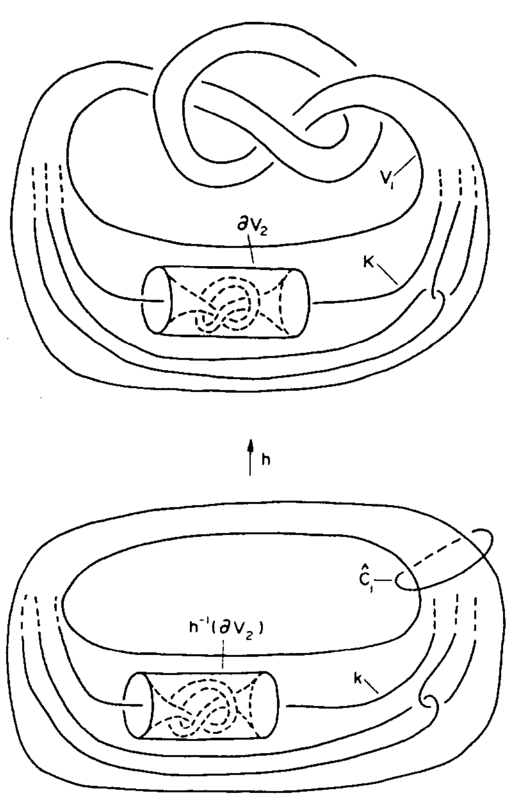

Fig. 3.

4. There is a solid torus W containing K which lies in $V_1 \cap V_2$ in such a way that its core meets some meridian disc of each V_i once only.

Since the complements of V_1 and V_2 are homeomorphic it follows in cases 1 and 2 that their boundaries are parallel, giving the required homeomorphism, otherwise an infinite sequence of disjoint non-parallel incompressible tori could be constructed in the complement of V_1, contradicting [2, p. 140].

In case 4 there will be a homeomorphism carrying V_1 to V_2 and fixing W. For the complement of W in V_i is homeomorphic to the complement of a link which consists of a knot l_i and a meridian of l_i, while the complement of W in S^3 is the complement of $l_i \# l$. Thus $l_1 = l_2$, and the homeomorphism is readily constructed.

This leaves only case 3 to consider. Let h be the faithful homeomorphism from the complement of an open neighbourhood of the closed braid $\hat{C}_1$ to the solid torus V_1, with $h(k) = K$, where k is the axis of $\hat{C}_1$. The knotted complement of V_2 lies in $V_1 \backslash K$, so its inverse image under h forms the side of the torus $h^{-1}(\partial V_2)$ in S^3 which does not contain k. (See Fig. 3).

The unknotted curve k in S^3 must then lie in a solid turus bounded by $h^{-1}(\partial V_2)$, whose complement is homeomorphic to the complement of the knot l. It follows that k must lie in a ball inside the solid torus, otherwise k would be knotted, with l as a companion. Thus k has linking number 0 with every curve on $h^{-1}(\partial V_2)$.

Since a faithful homeomorphism preserves linking number, the curve K must have linking number 0 with every curve on ∂V_2. But, by construction of V_2, a meridian curve on ∂V_2 will have linking number n, the number of strings in C_2, with K. Hence case 3 cannot occur, and Theorem 2 is established.

REFERENCES

1. J. S. Birman: Braids, links, and mapping class groups. *Ann. Math. Stud.* **82** (1974).
2. J. Hempel: 3-Manifolds. *Ann. Math. Stud.* **86** (1976).
3. L. P. Neuwirth: The status of some problems related to knot groups. Topology conference 1973, *Springer Lecture Notes No.* 375.
4. H. Schubert: Knoten und Vollringe. *Acta Math.* **90** (1953), 131–286.
5. W. Whitten: Fibered knots through T-surgery. *Proc. Am. math. Soc.* **34** (1972), 293–298.

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